EPH - International Journal of Science And Engineering
ISSN (Online): 2454 2016
Volume 07 Issue 03 September 2021

DOI: https://doi.org/10.53555/ephijse.v7i3.189

A BIFURCATION STUDY: EFFECT OF A TOXICANT ON A BIOLOGICAL SPECIES, EMITTED BY ITSELF IN ITS OWN ENVIRONMENT

A.K. Agrawal[1]*, Anuj Kumar[2]
*[1]*Department of Mathematics, Amity University, Lucknow, India*
[2]*Department of Mathematics, Integral University, Lucknow, India*

***Corresponding Author:-**
[1]*Email: akagrawal@lko.amity.edu*
[2]*Email: anuj.kumaragarwal20@gmail.com*

Abstract*:*
In this paper, a model based on effect of toxicant on a biological species [1, chap. 2] is analyzed for the existence and nature of hopf-bifurcation. The biological population is logistically growing in its own environment and toxicant is being emitted by the population itself. The hopf-bifurcation analysis of model shows that when the emission rate of toxicant by the biological population increases in the environment, the density level of biological population decreases and after crossing a critical value of emission rate, the density of biological population starts oscillating and never settle down to its equilibrium level. The hopf-bifurcation analysis of model increases the validity of model. The dynamic behavior of the model for the emission rate of toxicant is described by providing numerical simulation.

Keywords: *Biological population, mathematical model, toxicant, supercritical hopfbifurcation.*

***AMS Classification –** 34C23, 34C25, 37L10, 37N25, 92D25*

I. INTRODUCTION

In the recent decades, use of mathematical models is increasing in various fields such as ecology, epidemiology, ecotoxicology, etc. Ecotoxicology is a field in which we study the effect of toxicants on biological species. Many studies have been carried out in the field of ecotoxicology using mathematical models [1, 2, 3, 5, 7, 8, 11 and 12]. Agrawal [1, chap. 2] proposed a mathematical model to study the effect of a toxicant on a biological species in which toxicant is discharged by the biological species itself in its own environment and decreases the growth rate of biological species as well as carrying capacity of biological species in the environment. It is a well-defined model. Local stability analysis and global stability analysis of the model are analyzed using Routh-hurwitz criterion and Lyapunov's function which characterizes the stable behavior of mathematical model very well. But a very important part of analysis (i.e. bifurcation analysis) which describe the unstable behavior of mathematical model and increase the validity of the model, is left there for further study.

II. Model description

The model for the effect of toxicant on a logistically growing biological population by Agrawal [1, chap. 2] is given as follows:

$$\begin{aligned}
\frac{dN}{dt} &= \left[r(U) - \frac{r_0 N}{K(T)}\right] N \\
\frac{dT}{dt} &= \lambda N - \delta_0 T - \alpha N T + \pi \nu N U \qquad (2.1)\\
\frac{dU}{dt} &= -\delta_1 + \alpha N T - \nu N U
\end{aligned}$$

$$N(0) = N_0 \geq 0, \quad T(0) = T_0 \geq 0, \quad U(0) = cN(0), \quad c \geq 0, \quad 0 \leq \pi \leq 1$$

Here, (t) is the density of the biological population, $T(t)$ is the concentration of the toxicant in the environment and $U(t)$ is the uptake concentration of toxicant by the population $N(t)$. All the parameters defined in model are positive. r_0 is the growth rate of biological species in a toxic-free environment. λ is the emission rate of toxicant in its own environment by the biological population. δ_0 is the natural washout rate coefficient of (t), α is the depletion rate coefficient of $T(t)$ due to uptake by population, δ_1 is the natural washout rate coefficient of $U(t)$. The constant ν is the depletion rate coefficient of (t) due to decay of some members of $N(t)$ and a fraction π re-enter into the environment. The constant $c \geq 0$ is the proportionality constant determining the measure of initial toxicant concentration in the population at $t = 0$. The function (U) represents the growth rate of biological depending on the uptake of toxicant by the biological population. (T) is the carrying capacity function for the biological population depend on the emission of toxicant of biological population $T(t)$. (T) decreases when the concentration level of toxicant $T(t)$ increases in the environment and vice-versa. The model (2.1) has two nonnegative equilibrium points, $E_1(0, 0, 0)$ and $E_2(N^*, T^*, U^*)$. All the positive solutions of model must lie in the region Ω, (see Agrawal [1, chap. 2]) where,

$$\Omega = \left\{(N, T, U):\ 0 \leq N \leq K_0,\ 0 \leq T + U \leq \frac{\lambda K_0}{\delta}\right\}, \text{ here } \delta = \min(\delta_0, \delta_1)$$

III. Hopf-bifurcation Analysis

Hopf-bifurcation analysis [6, 9 and 10] is a very important part to describe the qualitative behavior of mathematical model. It is well known, a model system based on nonlinear ordinary differential has a hopf-bifurcation, if variational matrix corresponding to the equilibrium point has two purely imaginary eigenvalues and other eigenvalues have negative real parts.

In this section, we analyzed model (2.1) for the existence of Hopf-bifurcation corresponding to the equilibrium point E_2 by taking λ (i.e. emission rate of toxicant by the bioloigical population) as a bifurcation parameter. We linearize the model (2.1) about equilibrium point's E_2 by using the following transformation:

$$N = N^* + n, \qquad T = T^* + \tau, \qquad U = U^* + u$$

where n, τ and u are small perturbation around E_2.

The matrix form of model (2.1) in terms of variables n, τ and u can be written as,

$$\dot{X} = AX + B \qquad (3.1)$$

where

$$X = \begin{bmatrix} n \\ \tau \\ u \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Moreover,

$$a_{11} = -\frac{r_0 N^*}{K(T^*)}, \qquad a_{12} = \frac{r_0 N^{*2}}{K^2(T^*)} K'(T^*), \qquad a_{13} = r'(U^*) N^*,$$

$$a_{21} = \lambda - (\alpha T^* - \pi \nu U^*), \qquad a_{22} = -(\delta_0 + \alpha N^*), \qquad a_{23} = \pi \nu N^*,$$

$$a_{31} = \alpha T^* - \nu U^*, \qquad a_{32} = \alpha N^*, \qquad a_{33} = -(\delta_1 + \nu N^*)$$

and

$$b_1 = r'(U^*).nu + 2\frac{r_0 N^* K'(T^*)}{K^2(T^*)}.n\tau + \frac{r_0 K'(T^*)}{K^2(T^*)}.n^2\tau - \frac{r_0}{K(T^*)}.n^2$$

$$b_2 = \pi\nu.nu - \alpha.n\tau$$

$$b_3 = \alpha.n\tau - \nu.nu$$

Here, Ax is linear and B is nonlinear part of model (2.1). Matrix A is similar to the variational matrix M_2 corresponding to the equilibrium point E_2 (see [1, chap. 2]). The characteristic equation of matrix A can be written as:

$(x) = x^3 + c_1 x^2 + c_2 x + c_3$ (3.2)

Where

$c1 = -(a11 + a22 + a33)$

$c2 = a11a22 + a11a33 + a22a33 - a12a21 - a13a31 - a23a32$

$c3 = a11a23a32 + a22a13a31 + a33a12a21 - a11a22a33 - a12a23a31 - a13a21a32$

The characteristic equation (3.2) of model (2.1) is of order 3, so there are three eigenvalues of matrix A and one eigenvalue of matrix must be real (i.e. $x_{1,2} = R_1 \pm iI_1$ and $x_3 = R_3$). So, the model (2.1) undergoes a hopf-bifurcation corresponding to the equilibrium point E_2, if eq.(3.2) have two purely imaginary complex conjugate eigenvalues and a negative eigenvalue (i.e. $R_1 = 0$ and $R_3 < 0$) at $\lambda = \lambda^*(> 0)$ and $\frac{dR_1}{d\lambda} \neq 0$ at $\lambda = \lambda^*$.

Now, according to the Liu's criterion [10], model undergoes a Hopf-bifurcation at $\lambda = \lambda^* > 0$, if the following conditions hold:

- $A_j(\lambda^*) > 0, \quad j = 1,3$
- $A_1(\lambda^*)A_2(\lambda^*) - A_3(\lambda^*) = 0$
- At $\lambda = \lambda^*$, the derivative of the real part of complex conjugate eigenvalue with respect to the parameter λ is not equal to zero. (i.e. $\frac{dR_1}{d\lambda} \neq 0$)

Now, we will verify the third condition of the existence of hop-bifurcation by putting

$x = R + iI$ in eq. (3.2), we get

$(R + iI)^3 + c_1 (R + iI)^2 + c_2(R + iI) + c_3 = 0$ (3.3)

Separating the real and imaginary parts, we get

$$R^3 - 3RI^2 + c_1R^2 - c_1I^2 + c_2R + c_3 = 0 \qquad (3.4)$$

$$-I^3 + 3R^2I + 2c_1RI + c_2I = 0 \qquad (3.5)$$

Eliminating I between eq. (3.4) and eq. (3.5), we have

$$-8R^3 - 8c_1R^2 - 2c_2R - 2c_1R^2 - c_1c_2 + c_3 = 0 \qquad (3.6)$$

Differentiating eq. (3.6) with respect to λ at λ^*,

$$\left[\frac{dR}{d\lambda}\right]_{\lambda=\lambda^*} = \left[\frac{\frac{d}{d\lambda}(c_1c_2 - c_3)}{-2(c_1^2 + c_2)}\right]_{\lambda=\lambda^*} \neq 0 \qquad (3.7)$$

So, we can state the following theorem.

Theorem 1 *The model (2.1) undergoes a Hopf-bifurcation from the equilibrium point E_2, if λ passes through the criticalvalue $\lambda^* > 0$ such that:*

1. $A_j(\lambda^*) > 0, \quad j = 1,3$
2. $A_1(\lambda^*)A_2(\lambda^*) - A_3(\lambda^*) = 0$
3. $\left[\frac{dR}{d\lambda}\right]_{\lambda=\lambda^*} = \left[\frac{\frac{d}{d\lambda}(c_1c_2-c_3)}{-2(c_1^2+c_2)}\right]_{\lambda=\lambda^*} \neq 0$

Now, it is important to know the nature of bifurcating periodic solutions arising through Hopf-bifurcation. We have analyzed the stability and direction of the bifurcating periodic solutions by following the procedure given by Hassard et al. (see [6]). At the Hopfbifurcation point, eq. (3.2) have a pair of complex conjugate roots of eq.(3.2) with zero real parts and a negative real root, so we can assume the roots such as:

$x_{1,2} = \pm iI, \qquad x_3 = -J$

Where, $I = \sqrt{c_2},\ J = -c_1$

Now, to determine the direction of Hopf-bifurcation and stability of bifurcating periodic solution, we transform the system (2.1) in normal form. Let $X = PY$ then the matrix form (3.1) of model becomes

$$Y' = JY + F, \qquad Y = col.\,(y_1, y_2, y_3) \tag{3.8}$$

Where

$$J = P^{-1}AP = \begin{bmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & -J \end{bmatrix} \quad \text{and} \quad F = P^{-1}f = \begin{bmatrix} F_1(y_1, y_2, y_3) \\ F_2(y_1, y_2, y_3) \\ F_3(y_1, y_2, y_3) \end{bmatrix} \tag{3.9}$$

Here P is a transformed matrix defined as

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

Where,

$P11 = a12a23a31 + a13a21a32 - a13a31a22 - a12a21a33$

$P12 = (a12a21 + a13a31)$

$$P_{13} = a_{12}(a_{23}a_{31} - a_{21}a_{33} - a_{21}J) + a_{13}(a_{21}a_{32} - a_{31}a_{22} - a_{31}J)$$

$$P_{21} = a_{11}(a_{21}a_{33} - a_{23}a_{31}) - a_{21}v^2, \qquad P_{22} = (a_{23}a_{31} - a_{11}a_{21} - a_{21}a_{33})v$$

$$P_{23} = (a_{11} + J)\{(a_{21}a_{33} - a_{23}a_{31}) + a_{21}J\}$$

$$P_{31} = a_{11}(a_{31}a_{22} - a_{21}a_{32}) - a_{31}v^2, \qquad P_{32} = (a_{21}a_{32} - a_{31}a_{11} - a_{31}a_{22})v$$

$$P_{33} = (a_{11} + J)\{(a_{22}a_{31} - a_{21}a_{32}) + a_{31}J\}$$

and

$$f = \begin{bmatrix} f_1(y_1, y_2, y_3) \\ f_2(y_1, y_2, y_3) \\ f_3(y_1, y_2, y_3) \end{bmatrix}$$

where

$$f1(y1, y2, y3) = r'(U*).\,(P11y1 + P12y2 + P13y3)(P31y1 + P32y2 + P33y3)$$
$$+ 2\frac{r_0N^*K'(T^*)}{K^2(T^*)}.\,(P_{11}y_1 + P_{12}y_2 + P_{13}y_3)(P_{21}y_1 + P_{22}y_2 + P_{23}y_3)$$
$$+ \frac{r_0K'(T^*)}{K^2(T^*)}.\,(P_{11}y_1 + P_{12}y_2 + P_{13}y_3)^2(P_{21}y_1 + P_{22}y_2 + P_{23}y_3)$$
$$- \frac{r_0}{K(T^*)}.\,(P_{11}y_1 + P_{12}y_2 + P_{13}y_3)^2$$

$$f_2(y_1, y_2, y_3) = \pi v.\,(P_{11}y_1 + P_{12}y_2 + P_{13}y_3)(P_{31}y_1 + P_{32}y_2 + P_{33}y_3) - \alpha.\,(P_{11}y_1 + P_{12}y_2 + P_{13}y_3)(P_{21}y_1 + P_{22}y_2 + P_{23}y_3)$$

$$f_3(y_1, y_2, y_3) = \alpha.\,(P_{11}y_1 + P_{12}y_2 + P_{13}y_3)(P_{21}y_1 + P_{22}y_2 + P_{23}y_3) - v.\,(P_{11}y_1 + P_{12}y_2 + P_{13}y_3)(P_{31}y_1 + P_{32}y_2 + P_{33}y_3)$$

Hence, eq. (3.8) is the normal form of eq. (2.1). For evaluating the direction of bifurcating solution, we evaluate following quantities at $\lambda = \lambda^*$ and $(y_1, y_2, y_3) = (0,0,0)$.

$$g_{11} = \frac{1}{4}\left\{\left(\frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_1}{\partial y_2^2}\right) + i\left(\frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2}\right)\right\},$$

$$g_{02} = \frac{1}{4}\left\{\left(\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} - 2\frac{\partial^2 F_2}{\partial y_1 \partial y_2}\right) + i\left(\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} + 2\frac{\partial^2 F_1}{\partial y_1 \partial y_2}\right)\right\}$$

$$g_{20} = \frac{1}{4}\left\{\left(\frac{\partial^2 F_1}{\partial y_1^2} - \frac{\partial^2 F_1}{\partial y_2^2} + 2\frac{\partial^2 F_2}{\partial y_1 \partial y_2}\right) + i\left(\frac{\partial^2 F_2}{\partial y_1^2} - \frac{\partial^2 F_2}{\partial y_2^2} - 2\frac{\partial^2 F_1}{\partial y_1 \partial y_2}\right)\right\},$$

$$g_{21} = G_{21} + 2G_{110}^1 w_{11}^1 + G_{101}^1 w_{20}^1$$

where,

$$G_{21} = \frac{1}{8}\left\{\left(\frac{\partial^3 F_1}{\partial y_1^3} + \frac{\partial^3 F_2}{\partial y_2^3} + \frac{\partial^3 F_2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 F_1}{\partial y_1 \partial y_2^2}\right) + i\left(\frac{\partial^3 F_2}{\partial y_1^3} - \frac{\partial^3 F_1}{\partial y_2^3} - \frac{\partial^3 F_1}{\partial y_1^2 \partial y_2} + \frac{\partial^3 F_2}{\partial y_1 \partial y_2^2}\right)\right\},$$

$$G_{110}^1 = \frac{1}{2}\left\{\left(\frac{\partial^2 F_1}{\partial y_1 y_3} + \frac{\partial^2 F_2}{\partial y_2 y_3}\right) + i\left(\frac{\partial^2 F_2}{\partial y_1 y_3} - \frac{\partial^2 F_1}{\partial y_2 y_3}\right)\right\},$$

$$G_{101}^1 = \frac{1}{2}\left\{\left(\frac{\partial^2 F_1}{\partial y_1 y_3} - \frac{\partial^2 F_2}{\partial y_2 y_3}\right) + i\left(\frac{\partial^2 F_2}{\partial y_1 y_3} + \frac{\partial^2 F_1}{\partial y_2 y_3}\right)\right\},$$

$$h_{11}^1 = \frac{1}{4}\left(\frac{\partial^2 F_3}{\partial y_1^2} + \frac{\partial^2 F_3}{\partial y_2^2}\right), \qquad h_{20}^1 = \frac{1}{4}\left(\frac{\partial^2 F_3}{\partial y_1^2} - \frac{\partial^2 F_3}{\partial y_2^2} - 2i\frac{\partial^2 F_3}{\partial y_1 \partial y_2}\right),$$

$$w_{11}^1 = \frac{-h_{11}^1}{J}, \qquad w_{20}^1 = \frac{-h_{20}^1}{J - 2iI},$$

To examine the nature of hopf-bifurcation and periodic solutions, we calculate the following quantities:

$$\left\{\begin{array}{ll} C_1(0) = \frac{i}{2I}\left[g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right] + \frac{1}{2}g_{21}, & \mu_2 = -\frac{Re\, C_1(0)}{Re\, x'(\lambda^*)}, \\ \beta_2 = 2Re\, C_1(0), & \tau_2 = -\frac{\left(Im\, C_1(0) + \mu_2 Im\, x'(\lambda^*)\right)}{I} \end{array}\right\} \quad (3.10)$$

After calculating above, we can state the following results:

Theorem 2 *Model (2.1) shows a Hopf-bifurcation corresponding to the parameter* λ *as follows:*

1. *If* $\mu_2 > 0$ *(or* $\mu_2 < 0$*), the Hopf-bifurcation is supercritical (or subcritical) and the bifurcating periodic solutions exist for* $\lambda > \lambda^*$ $(\lambda < \lambda^*)$,
2. *If* $\beta_2 < 0$ *(or* $\beta_2 > 0$*), the bifurcating periodic solutions are stable (or unstable),*
3. *If* $\tau_2 > 0$ *(or* $\tau_2 < 0$*), the period of bifurcating periodic solutions increases (or decreases).*

IV. Numerical simulation

In the previous section, we have found conditions for existence of hopf-bifurcation and its nature. To numerically clarify that model (2.1) has a supercritical bifurcation, we assume functions (U) and $K(T)$ as follows:

$$r(U) = r_0 - r_1 U \quad \text{and} \quad K(T) = K_0 - \frac{b_1 T}{1 + b_2 T} \quad (4.1)$$

and define the value of parameters:

$$\begin{array}{llll} r_0 = 0.2, & r_1 = 0.8, & K_0 = 10.0, & b_1 = 0.01, \\ b_2 = 1.0, & \lambda = 0.0007, & \delta_0 = 0.04, & \pi = 0.8, \\ \nu = 0.04, & \delta_1 = 0.0005, & \alpha = 0.001. & \end{array} \quad (4.2)$$

As, we assumed functions $r(U)$ and $K(T)$ in (4.1) and define the value of parameters in (4.2), we increase the value of parameter λ from 0.0007 to $\lambda^* = 0.38495$, we obtained that all the eigenvalues of variational matrix A have negative real parts ($R_1, R_2, R_3 < 0$) for each value of λ in this range but at λ^*, we get purely imaginary complex conjugate (*i. e.* $R_1 = R_2 = 0$ and $I_1 = -I_2 \neq 0$) and a negative real eigenvalue (*i. e.* $R_3 < 0$) of the variational matrix, (see Figs.1 & 2). It means model (2.1) undergoes a hopf-bifurcation at $\lambda = 0.38495$.

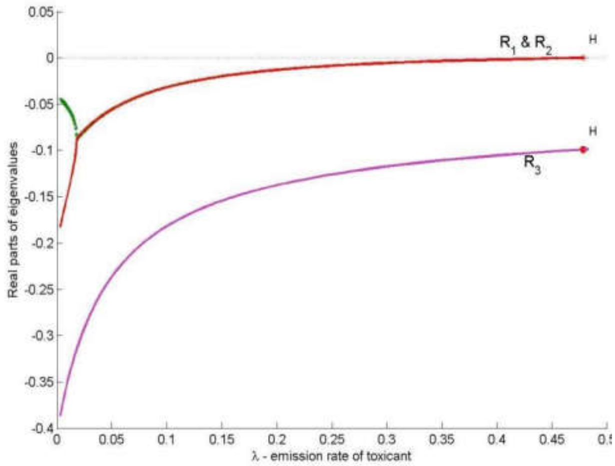

Figure 1: Real parts of eigenvalues of variational matrix A respect to the parameter λ

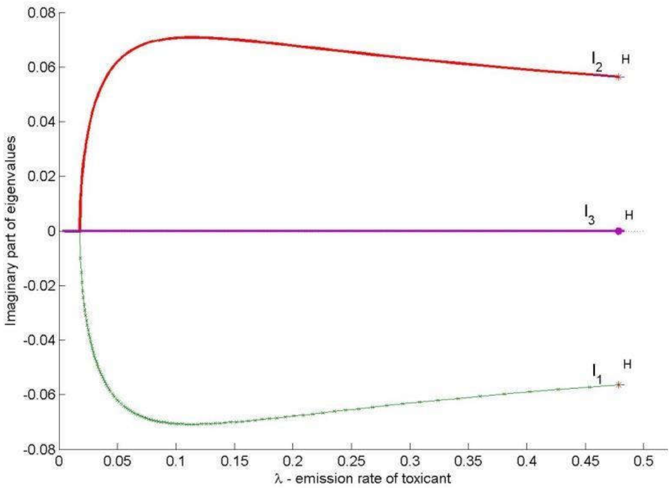

Figure 2: Imaginary parts of eigenvalues of variational matrix A respect to the parameter λ

In Fig.3, we have shown the density of biological population for different value of parameter λ with respect to time. This figure shows that when the value parameter $\lambda < 0.38495$, the density of biological population initially oscillates and then settle down to its equilibrium level. But when the value of parameter $\lambda > 0.38495$, the density of biological population oscillates and does not settle down to its equilibrium level. Here, it is obvious that there exist a hopf-bifurcation in the model (2.1) for parameter λ at 0.38495 and other value of parameters same as (4.2). Now, to find the nature of hopf-bifurcation we calculated the quantities $\mu_2 = 1.1630 * 10^{-5} > 0$ and $\beta_2 = -6.5744 * 10^{-7} < 0$. According to the

Theorem:2, there exist a supercritical hopf-bifurcation with stable bifurcating periodic solution. In Fig.4, we show the dynamic behavior of density of biological population N corresponding to λ. Here, we have shown that stable density level of biological population decreases when value of parameter λ increase upto $\lambda = 0.38459$. At the value of $\lambda = 0.38459$ system undergoes a supercritical hopf-bifurcation and after that we get an unstable equilibrium with stable periodic oscillations.

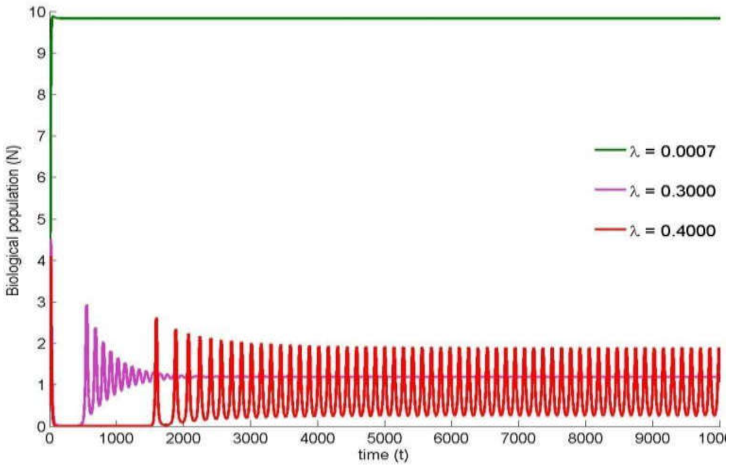

Figure 3: Variation of parameter λ with respect to time.

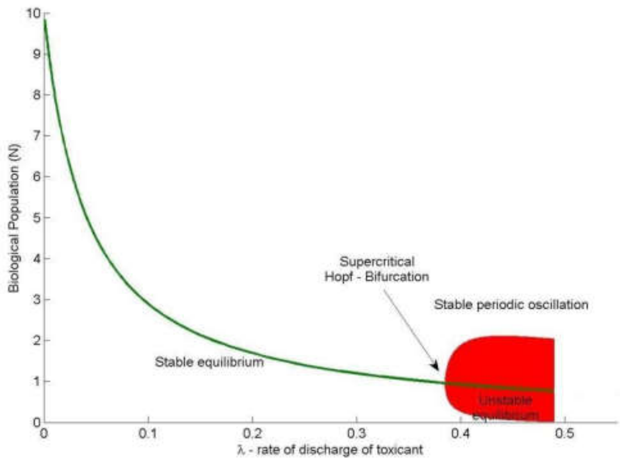

Figure 4: Dynamic behavior of density of biological population N respect to the parameter λ.

V. Conclusion

In this paper, we have analyzed a mathematical model based on effect of toxicant on a logistically growing biological population [1, chap. 2] for the existence and nature of hopfbifurcation. The hopf-bifurcation analysis of model (2.1) shows that parameter λ (i.e. the emission rate of toxicant by biological species itself) has a critical value λ^* such that model (2.1) has stable solutions at the equilibrium point $E_2(N^*, T^*, U^*)$for each value of $\lambda < \lambda^*$ and model undergoes a supercritical hopf-bifurcation at λ^*, after that we found unstable solution for each value of $\lambda > \lambda^*$. The hopf-bifurcation analysis of model show that for higher emission of toxicant in the environment by the biological species, density of biological population become unstable and never settles to its equilibrium level.

References

[1]. Agrawal, A.K., (1999). Effects of Toxicants on Biological Species: Some non-linear mathematical models and their analyses. *Ph. D. Thesis*, *Department of Mathematics, I.I.T. Kanpur, INDIA.*

[2]. Agrawal, A.K., & Shukla, J.B. (2012) Effect of a toxicant on a biological population causing severe symptoms on a subclass. *South Pacific Journal of Pure and Applied Mathematics*, 1(1):12–27.

[3]. Agrawal, A.K., Sinha, P., Dubey, B. & Shukla, J.B. (2000). Effects of two or more toxicants on a biological species: A non-linear mathematical model and its analysis. *In Mathematical Analysis and Applications. A.P. Dwivedi (Ed)*, Narosa Publishing House, New Delhi, INDIA: 97 – 113.

[4]. Dhooge, A., Govaerts, W., & Kuznetsov, Y.A. (2003). MATCONT: a MATLAB package for numerical bifurcation analysis of ODEs. *ACM Transactions on Mathematical Software (TOMS)*, 29(2), 141-164.

[5]. Freedman, H.I. & Shukla J.B. (1991). Models for the effect of toxicant in single species and predator prey systems. *J. Math. Biol.*, 30:15 – 30.

[6]. Hassard, B.D., Kazarinoff, N.D. & Wan, Y. (1981). Theory and Application of Hopf Bifurcation. *London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, UK*, 41.

[7]. Kumar, A., Agrawal, A.K., Hasan, A. & Misra, A.K. (2016). Modeling the effect of toxicant on the deformity in a subclass of a biological species. *Model. Earth Syst. Environ.*, 2(1).
[8]. Kumar, A., Khan, A.W. & Agrawal A.K. (2015). Modeling the simultaneous effect of two toxicants causing deformity in a subclass of biological population. *IOSR Journal of Mathematics*, 11(6):70–82.
[9]. Kuznetsov, Y. (2004). Elements of Applied Bifurcation Theory. *NewYork: Springerverlag*.
[10]. Liu, W. (1994). Criterion of Hopf Bifurcation without using Eigenvalues. *J. Math. Anal. and App.*, 182:250–256.
[11]. Shukla, J.B. & Agrawal, A.K. (1999). Some mathematical models in ecotoxicology; Effects of toxicants on biological species. *Sadhana*, 24:25 – 40.
[12]. Shukla, J.B. & Dubey, B. (1996). Simultaneous effects of two toxicants on biological species: A mathematical model. *J. Biol. Systems*, 4:109 – 130.